Received: *June 19, 2001*, Accepted: *June 20, 2001*
Revised: *June 20, 2001*

A cohomological approach to the non-abelian Seiberg-Witten map

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Abstract: We present a cohomological method for obtaining the non-abelian Seiberg-Witten map for any gauge group and to any order in θ. By introducing a ghost field, we are able to express the equations defining the Seiberg-Witten map through a coboundary operator, so that they can be solved by constructing a corresponding homotopy operator.

Keywords: String Duality, Brane Dynamics in Gauge Theories, Gauge Symmetry, Non-Commutative Geometry.

JHEP06(2001)047

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1. Introduction

In the last few years noncommutative geometry has found physical realizations in string theory as argued originally in [1]. Based on the existence of different regularization procedures in string theory, Seiberg and Witten [2] claimed that certain noncommutative gauge theories are equivalent to commutative ones. In particular, they argued that there exists a map from a commutative gauge field to a noncommutative one, which is compatible with the gauge structure of each. This map has become known as the Seiberg-Witten (SW) map. In this paper, we give a method for explicitly finding this map. We will consider gauge theories on the noncommutative space defined by

$$\left[x^i \overset{\star}{,} x^j\right] = i\theta^{ij}\,, \tag{1.1}$$

where θ is a constant Poisson tensor. Then the "$\star$" operation is the associative Weyl-Moyal product

$$f \star g = f\, e^{\frac{i}{2}\theta^{ij}\overleftarrow{\partial_i}\overrightarrow{\partial_j}} g\,. \tag{1.2}$$

We believe that our methods are much more general, and can in fact be used even when θ is not constant.

In the next section, we review the methods developed in [3], which provide an essential starting point for our work. In section 3 we replace the gauge parameters appearing in the SW map with a ghost field, which makes explicit a cohomological structure underlying the SW map. In section 4 we define a homotopy operator, which can be used to explicitly write down the SW map order by order in θ. In section 5, we discuss some complications that arise in this formalism and some methods to overcome them. Finally, in section 6 we apply our methods to calculate some low order terms of the SW map. An appendix contains some expansions of the star product, that will be useful in the rest of the paper.

2. General review

In this section, we review the formalism developed in [3], which provides an alternative method for obtaining an expression for the SW map.

The original equation which defines the SW map [2] arises from the requirement that gauge invariance be preserved in the following sense. Let a_i, α be the gauge field and gauge parameter of the commutative theory and similarly let A_i, Λ be the gauge field and gauge parameter of the noncommutative theory. Under an infinitesimal gauge transformation,

$$\delta_\alpha a_i = \partial_i \alpha - i[a_i, \alpha]\,, \tag{2.1}$$

$$\delta_\Lambda A_i = \partial_i \Lambda - i[A_i \overset{\star}{,} \Lambda] \equiv \partial_i \Lambda - i\left(A_i \star \Lambda - \Lambda \star A_i\right). \tag{2.2}$$

Then, the SW map is found by requiring

$$A_i + \delta_\Lambda A_i = A_i(a_j + \delta_\alpha a_j, \ldots)\,. \tag{2.3}$$

In order to satisfy (2.3) the noncommutative gauge field and gauge parameter must have the following functional dependence.

$$\begin{aligned} A_i &= A_i(\, a, \partial a, \partial^2 a, \ldots) \\ \Lambda &= \Lambda(\, \alpha, \partial\alpha, \ldots, a, \partial a, \ldots)\,, \end{aligned} \tag{2.4}$$

where the dots indicate higher derivatives. It must be emphasized that a SW map is not uniquely defined by condition (2.3). The ambiguities that arise [10] will be discussed shortly.

The condition (2.3) yields a simultaneous equation for A_i and Λ. For the constant θ case, explicit solutions of the Seiberg-Witten map have been found by various authors up to second order in θ [11, 3]. The solutions were found by writing the map as a linear combination of all possible terms allowed by index structure and dimensional constraints and then determining the coefficients by plugging this expression into the SW equation. The method we will describe in the rest of the paper provides a more systematic procedure for solving the SW map. For the special case of a U(1) gauge group, an exact solution in terms of the Kontsevich formality map is given in [4], while [5]–[8] present an inverse of the SW map to all orders in θ.

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An alternative characterization of the Seiberg-Witten map can be obtained following [3]. In the commutative gauge theory, one may consider a field ψ in the fundamental representation of the gauge group. If we assume that the SW map can be extended to include such fields, then there will be a field Ψ in the noncommutative theory with the following functional dependence

$$\Psi = \Psi(\psi, \partial\psi, \ldots, a, \partial a, \ldots)\,, \tag{2.5}$$

and with the corresponding infinitesimal gauge transformation

$$\delta_\alpha \psi = i\alpha\psi \tag{2.6}$$
$$\delta_\Lambda \Psi = i\Lambda \star \Psi\,. \tag{2.7}$$

An alternative to the SW condition (2.3) can now be given by

$$\Psi + \delta_\Lambda \Psi = \Psi(\psi + \delta_\alpha\psi, \ldots, a_j + \delta_\alpha a_j, \ldots)\,. \tag{2.8}$$

More compactly, one writes

$$\delta_{\Lambda_\alpha} \Psi(\psi, a_j, \ldots) = \delta_\alpha \Psi(\psi, a_j, \ldots)\,. \tag{2.9}$$

The dependence of Λ on α is shown explicitly on the left hand side, and on the right hand side δ_α acts as a derivation on the function Ψ, with an action on the variables ψ and a_i given by (2.6) and (2.1) respectively. Next, one considers the commutator of two infinitesimal gauge transformations

$$\left[\delta_{\Lambda_\alpha}, \delta_{\Lambda_\beta}\right] \Psi = [\delta_\alpha, \delta_\beta]\,\Psi\,. \tag{2.10}$$

Since $[\delta_\alpha, \delta_\beta] = \delta_{-i[\alpha,\beta]}$, the right hand side of (2.10) can be rewritten as

$$\begin{aligned} \delta_{-i[\alpha,\beta]}\Psi &= \delta_{\Lambda_{-i[\alpha,\beta]}}\Psi \\ &= i\Lambda_{-i[\alpha,\beta]} \star \Psi = \Lambda_{[\alpha,\beta]} \star \Psi\,. \end{aligned}$$

The last equality follows from the fact that Λ is linear in the ordinary gauge parameter, which is infinitesimal. As for the left hand side,

$$\begin{aligned} \left[\delta_{\Lambda_\alpha}, \delta_{\Lambda_\beta}\right] \Psi &= \delta_{\Lambda_\alpha}\left(i\Lambda_\beta \star \Psi\right) - \delta_{\Lambda_\beta}\left(i\Lambda_\alpha \star \Psi\right) \\ &= i\left(\delta_\alpha\Lambda_\beta - \delta_\beta\Lambda_\alpha\right) \star \Psi + [\Lambda_\alpha \overset{\star}{,} \Lambda_\beta] \star \Psi\,. \end{aligned}$$

Equating the two expressions and dropping Ψ yields

$$(\delta_\alpha\Lambda_\beta - \delta_\beta\Lambda_\alpha) - i\,[\Lambda_\alpha \overset{\star}{,} \Lambda_\beta] + i\Lambda_{[\alpha,\beta]} = 0\,. \tag{2.11}$$

An advantage of this formulation is that (2.11) is an equation in Λ only, whereas (2.3) must be solved simultaneously in Λ and A_i. If (2.11) is solved, (2.2) then yields an equation for A_i and (2.7) for Ψ.

3. The ghost field and the coboundary operator

It is advantageous to rewrite equations (2.2), (2.7) and (2.11) in terms of a ghost field in order to make explicit an underlying cohomological structure. Specifically, we replace the gauge parameter α with a ghost v, which is an enveloping algebra valued, grassmannian field[1]. We define a ghost number by assigning ghost number one to v and zero to a_i and ψ. The ghost number introduces a Z_2 grading, with even quantities commuting and odd quantities anticommuting. In our formalism, the gauge transformations (2.1) and (2.6) are replaced by the following BRST transformations:

$$\begin{aligned} \delta_v v &= i v^2 \\ \delta_v a_i &= \partial_i v - i\,[a_i, v] \\ \delta_v \psi &= i v \psi\,. \end{aligned} \tag{3.1}$$

We also take δ_v to commute with the partial derivatives,

$$[\delta_v, \partial_i] = 0\,. \tag{3.2}$$

The operator δ_v has ghost number one and obeys a graded Leibniz rule

$$\delta_v(f_1 f_2) = (\delta_v f_1) f_2 + (-1)^{\deg(f_1)} f_1(\delta_v f_2)\,, \tag{3.3}$$

where $deg(f)$ gives the ghost number of the expression f. One can readily check that δ_v is nilpotent on the fields a_i, ψ and v and therefore, as a consequence of (3.3), we have

$$\delta_v^2 = 0\,. \tag{3.4}$$

Following the procedure outlined in the previous section, we characterize the SW map as follows. We introduce a matter field $\Psi(\psi, \partial\psi, \ldots, a, \partial a, \ldots)$ and an odd gauge parameter $\Lambda(v, \partial v, \ldots, a, \partial a, \ldots)$ corresponding to ψ and v in the commutative theory. Λ is linear in the infinitesimal parameter v and hence has ghost number one. As before, we require that the SW map respect gauge invariance.

$$\delta_\Lambda \Psi \equiv i\Lambda \star \Psi = \delta_v \Psi\,. \tag{3.5}$$

The consistency condition (2.10) now takes the form

$$\delta_\Lambda^2 \Psi = \delta_v^2 \Psi = 0\,, \tag{3.6}$$

and again it yields an equation in Λ only

$$0 = \delta_\Lambda^2 \Psi = \delta_\Lambda(i\Lambda \star \Psi) = i\delta_v \Lambda \star \Psi + \Lambda \star \Lambda \star \Psi\,.$$

[1] In the U(1) case the introduction of a ghost has been considered in [12].

Since Ψ is arbitrary we obtain

$$\delta_v \Lambda = i\Lambda \star \Lambda \,. \tag{3.7}$$

Once the solution of (3.7) is known, one can solve the following equations for Ψ and the gauge field.

$$\delta_v \Psi = i\Lambda \star \Psi \,, \qquad \delta_v A_i = \partial_i \Lambda - i \left[A_i \stackrel{\star}{,} \Lambda \right] \,. \tag{3.8}$$

It is natural to expand Λ and A_i as power series in the deformation parameter θ. We indicate the order in θ by an upper index in parentheses

$$\begin{aligned} \Lambda &= \sum_{n=0}^{\infty} \Lambda^{(n)} = v + \sum_{n=1}^{\infty} \Lambda^{(n)} \\ A_i &= \sum_{n=0}^{\infty} A_i^{(n)} = a_i + \sum_{n=1}^{\infty} A_i^{(n)} \,. \end{aligned} \tag{3.9}$$

Note that the zeroth order terms are determined by requiring that the SW map reduce to the identity as θ goes to zero. Using this expansion we can rewrite equations (3.7) and (3.8) as

$$\delta_v \Lambda^{(n)} - i\{v, \Lambda^{(n)}\} = M^{(n)} \tag{3.10}$$

$$\delta_v A_i^{(n)} - i[v, A_i^{(n)}] = U_i^{(n)} \,, \tag{3.11}$$

where, in the first equation, $M^{(n)}$ collects all terms of order n which do not contain $\Lambda^{(n)}$,and similarly $U_i^{(n)}$ collects terms not involving $A_i^{(n)}$. We refer to the left hand side of each equation as its homogeneous part, and to M and U_i as the inhomogeneous terms of (3.10) and (3.11). Note that $M^{(n)}$ contains explicit factors of θ, originating from the expansion of the Weyl-Moyal product (1.2). An expression for the generic $M^{(n)}$ is given in the Appendix. If the SW map for Λ is known up to order $(n-1)$, then $M^{(n)}$ can be calculated explicitly as a function of v and a_i. On the other hand, $U^{(n)}$ depends on both Λ and A_i, the former up to order n and the latter up to order $(n-1)$. Still, one can calculate it iteratively as a function of v and a_i.

The structure of the homogeneous portions suggests the introduction of a new operator Δ.

$$\Delta = \begin{cases} \delta_v - i\{v, \cdot\} & \text{on odd quantities} \\ \delta_v - i[v, \cdot] & \text{on even quantities} \end{cases} \tag{3.12}$$

In particular, Δ acts on v and a_i as follows.

$$\Delta v = -iv^2 \,, \qquad \Delta a_i = \partial_i v \,. \tag{3.13}$$

As a consequence of its definition, Δ is an anti-derivation with ghost-number one. It follows a graded Leibniz rule identical to the one for δ_v (3.3). Another consequence of the definition (3.12) is that Δ is nilpotent

$$\Delta^2 = 0 \,. \tag{3.14}$$

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The action of Δ on expressions involving a_i and its derivatives can also be characterized in geometric terms. Specifically, Δ differs from δ_v in that it removes the covariant part of the gauge transformation. Therefore, Δ acting on any covariant expression will give zero. For instance, if one constructs the field-strength, $F_{ij} \equiv \partial_i a_j - \partial_j a_i - i[a_i, a_j]$, one finds by explicit calculation

$$\Delta F_{ij} = 0\,. \tag{3.15}$$

It can also be checked that the covariant derivative, $D_i = \partial_i - i[a_i, \cdot]$ commutes with Δ,

$$[\Delta, D_i] = 0\,. \tag{3.16}$$

In terms of Δ the equations (3.10) and (3.11) take the form

$$\Delta\Lambda^{(n)} = M^{(n)} \tag{3.17}$$

$$\Delta A_i^{(n)} = U_i^{(n)}\,. \tag{3.18}$$

In the next section we will provide a method to solve these equations. Also note that since $\Delta^2 = 0$, it must be true that

$$\begin{aligned} \Delta M^{(n)} &= 0 \\ \Delta U_i^{(n)} &= 0\,. \end{aligned} \tag{3.19}$$

Indeed one should verify that (3.19) holds order by order. If (3.19) did not hold, this would signal an inconsistency in the SW map.

4. The homotopy operator

For simplicity, we begin by considering in detail the SW Map for the case of the gauge parameter Λ. Much of what we say actually applies to the other cases as well with minor modifications.

In the previous section, we have seen that order by order in an expansion in θ, the SW map has the form:

$$\Delta\Lambda^{(n)} = M^{(n)}\,, \tag{4.1}$$

where $M^{(n)}$ depends only on $\Lambda^{(i)}$ with $i < n$. Clearly, if one could invert Δ somehow, we could solve for Λ. But Δ is obviously not invertible, as $\Delta^2 = 0$. In particular, the solutions of (4.1) are not unique, since if Λ is a solution so is $\Lambda + \Delta S$ for any S of ghost number zero.[2] That is, Δ acts like a coboundary operator in a cohomology theory, and the solutions that we are looking for are actually cohomology classes of solutions, unique only up to the addition of Δ-exact terms. The formal existence

[2] These are precisely the ambiguities in the SW map that were first discussed in [10], where our operator Δ was called $\hat{\delta}'$.

of the SW Map is then equivalent to the statement that the cycle $M^{(n)}$ is actually Δ-exact for all n. Since we know that $\Delta^2 = 0$, this fact would follow as a corollary of the stronger statement that there is no non-trivial Δ-cohomology in ghost number two. In other words, there are no Δ-closed, order n polynomials with ghost number two which are not also Δ-exact. To prove this stronger claim, we could proceed as follows. Suppose that we could construct an operator K such that

$$K\Delta + \Delta K = 1\,. \tag{4.2}$$

Clearly, K must reduce ghost number by one, and therefore must be odd. Consider its action on a cycle M, (so $\Delta M = 0$)

$$(K\Delta + \Delta K)M = \Delta K M = M\,. \tag{4.3}$$

Therefore, $M = \Delta\Lambda$, with $\Lambda = KM$, which not only shows that M is exact, but also computes explicitly a solution to the SW map. We note that this method of solution is nearly identical to the method used by Stora and Zumino [9] to solve the Wess-Zumino consistency conditions for non-abelian anomalies. In fact, it was the parallels between these problems that motivated our current approach.

We now proceed to construct K. First we notice that $M^{(n)}$ depends on v only through its derivative $\partial_i v$, as one can see by looking at the explicit expressions in the appendix. The same is true for $U_i^{(n)}$ since it depends on v only through Λ. It is convenient to define

$$b_i = \partial_i v\,, \tag{4.4}$$

so that M and U_i can all be expressed as functions of a_i, b_i and their derivatives only. Furthermore, we rewrite $M^{(n)}$ solely in terms of covariant derivatives, rather than ordinary ones. After these replacements, we may consider $M^{(n)}$ an element of the algebra generated by a_i, b_i, and D_i. As explained in the next section this algebra is not free, but for the moment we ignore this issue. The action of the operator Δ takes on a particularly simple form in terms of these variables:

$$\Delta a_i = b_i\,, \quad \Delta b_i = 0\,, \qquad [\Delta, D_i] = 0\,. \tag{4.5}$$

Thus, it is natural to define K on these variables. A natural guess is

$$K a_i = 0\,, \qquad K b_i = a_i\,. \tag{4.6}$$

Since K inverts an operator which acts like a graded derivation, it cannot itself obey the Leibniz rule. We can instead proceed by defining an infinitesimal form of the operator K, which does. In particular, to define K, we first define two operators ℓ and δ such that

$$\Delta\ell + \ell\Delta = \delta\,, \tag{4.7}$$

and then an operator T (a kind of integration operator) such that

$$T\delta M = M\,, \qquad T(\ell M) = KM\,. \tag{4.8}$$

The operator δ is some infinitesimal variation of a_i and b_i, which can be integrated to the identity. It is also defined to commute with the covariant derivative

$$[\delta, D_i] = 0\,. \tag{4.9}$$

The action of ℓ is defined by

$$\ell a_i = 0\,, \qquad \ell b_i = \delta a_i\,, \qquad [\ell, D_i] = 0\,, \qquad [\ell, \delta] = 0\,, \qquad \ell^2 = 0\,. \tag{4.10}$$

Finally, the integration operator T acting on any expression is implemented via the following procedure:

1. Choose the fields to be linearly dependent on t and δ to be the infinitesimal variation with respect to t:

$$\begin{aligned} \delta a_i &\to a_i dt \\ \delta b_i &\to b_i dt \\ a_i &\to t a_i \\ b_i &\to t b_i\,. \end{aligned} \tag{4.11}$$

   That is, we transform any expression,

$$N(a_i, b_i, \delta a_i, \delta b_i, D_i) \to N(ta_i, tb_i, a_i dt, b_i dt, D_i)\,. \tag{4.12}$$

2. Integrate from $t = 0$ to $t = 1$. Thus,

$$TN(a_i, b_i, \delta a_i, \delta b_i, D_i) = \int_0^1 N(ta_i, tb_i, a_i dt, b_i dt, D_i)\,. \tag{4.13}$$

Notice that this prescription requires that we rewrite any expression involving ordinary derivatives in terms of covariant derivatives and gauge fields only. We now show by induction that these definitions do in fact yield a homotopy operator K. It is easy to see that $\Delta\ell + \ell\Delta = \delta$ holds when acting on a_i or b_i alone. Suppose then that the equation holds when acting on two monomials f and g of order less than or equal to r in a_i and b_i. Then it follows that

$$(\Delta\ell + \ell\Delta)(fg) = ((\Delta\ell + \ell\Delta)f)g + f(\Delta\ell + \ell\Delta)g\,, \tag{4.14}$$

where all the cross terms have canceled out. By the induction hypothesis this expression is equal to $(\delta f)g + f\delta g$, which is just $\delta(fg)$. Thus $\Delta\ell + \ell\Delta = \delta$ holds on any monomial of degree greater than zero. Since this operator is distributive, (4.7) holds for any element of the algebra.

5. Constraints

We have so far only considered the free algebra, generated by a_i , b_i and D_i, where the construction of K was relatively simple. To show that our algebra is not free consider the following.

$$\begin{aligned} \Delta F_{ij} &= \Delta \left(D_i a_j - D_j a_i + i[a_i, a_j] \right) . \\ &= D_i b_j - D_j b_i + i[b_i, a_j] + i[a_i, b_j] . \end{aligned} \tag{5.1}$$

As an element of the free algebra, the left hand side is not zero, but according to (3.15) it should be. The problem becomes more serious when one rewrites $M^{(n)}$ in terms of the elements of the free algebra. Beyond first order, one finds the ΔM is no longer zero in general, but vanishes only by using the following constraints

$$[F_{ij}, \cdot\,] - i[D_i, D_j](\cdot) = 0 \,, \quad \Delta F_{ij} = 0 \,. \tag{5.2}$$

If $\Delta M^{(n)}$ is not zero identically, K no longer inverts Δ when acting on $M^{(n)}$, and we no longer have a method for solving (3.17) for $\Lambda^{(n)}$. The origin of the constraints can be traced to the fact that partial derivatives commute

$$\partial_i \partial_j - \partial_j \partial_i = 0 \,, \qquad \partial_i b_j - \partial_j b_i = 0 \,, \tag{5.3}$$

since $b_i = \partial_i v$. This is no longer manifest in our algebra. In fact, written in terms of covariant derivatives, (5.3) becomes (5.2). There seems to be no way to eliminate these constraints since K is not defined on v, but only on $b_i = \partial_i v$. One might expect that at higher orders one would have to use additional constraints to verify that $\Delta M^{(n)}$ vanishes, but this is not the case. For example, when one rewrites

$$\partial_i \partial_k b_j - \partial_j \partial_k b_i = 0 \tag{5.4}$$

in terms of covariant derivatives, the resulting expression is not an independent constraint, but can be written in terms of the two fundamental ones (5.2).

The reason why $\Delta M^{(n)}$ is not zero in general is because the existence of the constraints allows us to write $M^{(n)}$ in terms of the algebra elements in many different ways. Our goal will then be to define a procedure for writing $M^{(n)}$ in terms of algebra elements so that $\Delta M^{(n)} = 0$, identically. We will describe two procedures.

The first is the method we will use in the next section to calculate some low order terms of the SW map. We begin by obtaining an expression for $M^{(n)}$ in terms of the algebra elements. Generically, $\Delta M^{(n)}$ will be proportional to the constraints. At low orders, once $\Delta M^{(n)}$ is calculated, it is easy to guess an expression $m^{(n)}$, which is proportional to the constraints, such that the combination $M^{(n)} + m^{(n)}$ is annihilated by Δ. Acting K on this new combination then gives the solution $\Lambda^{(n)}$. We believe this guessing method can be formalized, but at higher orders we believe that the second procedure which we will now describe is simpler.

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First we introduce a new element of the algebra, f_{ij}, which is annihilated by all the operators defined in previous sections.

$$\Delta f_{ij} = \delta f_{ij} = \ell f_{ij} = 0\,. \tag{5.5}$$

We also introduce a new constraint

$$f_{ij} - F_{ij} = 0\,, \tag{5.6}$$

where F_{ij} is considered a function of D_i and a_i. We want to show that using this enlarged algebra and the constraints we can rewrite $M^{(n)}$ so that it has the following dependence.

$$M^{(n)} = M^{(n)}(a, b, (D^k a)_s, (D^l b)_s, D^h f)\,, \tag{5.7}$$

where the subscript s indicates that all the indices within the parentheses should be totally symmetrized. It would then follow that ΔM has the same functional dependence. Since it is impossible that ΔM contains any term antisymmetric in the indices of Da or Db, the constraints (5.6) and (5.2) cannot be generated. However, we may find that ΔM is proportional to the following constraints.

$$[f_{ij}, \cdot\,] - i[D_i, D_j](\cdot) = 0\ , \quad D_i f_{jk} + D_j f_{ki} + D_k f_{ij} = 0\,. \tag{5.8}$$

Since these constraints commute with the action of both K and Δ, if we add to M a term proportional to (5.8), our result for $\Lambda = KM$ is unchanged. To show that we can actually write M in the form suggested above, we begin with an expression for M as found by expanding the star product.

$$M^{(n)} = M^{(n)}(a, (\partial^k)_s a, (\partial^l)_s v)\,, \tag{5.9}$$

where we choose to explicitly write the derivatives in symmetric form. By replacing $\partial(\cdot) \to D(\cdot) + i[a, \cdot]$, and $\partial v \to b$ the expression takes the form

$$M^{(n)} = M^{(n)}(a, b, (D^k)_s a, (D^l b)_s)\,. \tag{5.10}$$

The difference $(D^k a)_s - D^k a$ contains terms that are proportional to the antisymmetric parts of DD or Da. But using the constraints we can make the following substitutions

$$[D_i, D_j](\cdot) \to -i[f_{ij}, \cdot\,]\,, \qquad D_i a_j - D_j a_i \to f_{ij} - i[a_i, a_j]\,. \tag{5.11}$$

This must be done recursively since the commutator term involving a's above may again be acted on by D's. But at each step, the number of possible D's acting on a is reduced by one. After carrying out this procedure M will have the form (5.7).

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6. Some calculations

In this final section, we use the formalism we have developed to compute some low order terms of the SW map. We focus mainly on solving for the gauge parameter Λ.

At the zeroth order, if we expand $\delta_v \Lambda = i\Lambda \star \Lambda$ we find

$$\delta_v v = i v^2 \tag{6.1}$$

which is just the BRST transformation of v (3.1). At first order, we have

$$\Delta\Lambda^{(1)} = -\frac{1}{2}\theta^{ij} b_i b_j \tag{6.2}$$

while at the second order we obtain

$$\Delta\Lambda^{(2)} = -\frac{i}{8}\theta^{ij}\theta^{kl}\partial_i b_k \partial_j b_l - \frac{1}{2}\theta^{ij}[b_i, \partial_j \Lambda^{(1)}] + i\Lambda^{(1)}\Lambda^{(1)} \,. \tag{6.3}$$

A solution of (6.2) has been found in [2] and is given by

$$\Lambda^{(1)} = \frac{1}{4}\theta^{ij}\{b_i, a_j\} \,. \tag{6.4}$$

We can reproduce this solution immediately by applying K to the expression $M^{(1)} = -\frac{1}{2}\theta^{ij} b_i b_j$. There are no problems at this level, since there are not enough derivatives for the constraints to show up. As explained in the previous sections we proceed in two steps. We first apply ℓ

$$\ell(M^{(1)}) = -\frac{1}{2}\theta^{ij}(\delta a_i b_j - b_i \delta a_j) \,, \tag{6.5}$$

then T to find

$$K(M^{(1)}) = \frac{1}{2}\theta^{ij}(b_i a_j + a_j b_i)\int_0^1 dt t = \frac{1}{4}\theta^{ij}\{b_i, a_j\} \,. \tag{6.6}$$

The ambiguity in the first order solution as determined in [10] is proportional to

$$\tilde{\Lambda}^{(1)} = -2i\theta^{ij}[b_i, a_j] \,. \tag{6.7}$$

According to the previous discussion the ambiguity amounts to an exact cocycle, hence is of the form:

$$\tilde{\Lambda}^{(1)} = \Delta S^{(1)} \,, \tag{6.8}$$

where $S^{(1)}$ can be computed to be

$$S^{(1)} = K\tilde{\Lambda}^{(1)} = -i\theta^{ij}[a_i, a_j] \,. \tag{6.9}$$

Solutions at the second order have been found by various authors. In [3] the following solution is presented,

$$\Lambda^{(2)} = \frac{1}{32}\theta^{ij}\theta^{kl}\Big(- \{b_i, \{a_k, i[a_j, a_l] + 4\partial_l a_j\}\} - i\{a_j, \{a_l, [b_i, a_k]\}\} + \\ + 2[[b_i, a_k] + i\partial_i b_k, \partial_j a_l] + 2i[[a_j, a_l], [b_i, a_k]]\Big) \,, \tag{6.10}$$

while in [11] the following solution is found,

$$\Lambda'^{(2)} = \frac{1}{32}\theta^{ij}\theta^{kl}\Big(- \{b_i, \{a_k, i[a_j, a_l] + 4\partial_l a_j\}\} - i\{a_j, \{a_l, [b_i, a_k]\}\} + \\ + 2[[b_k, a_i] + i\partial_i b_k, \partial_j a_l]\Big) \,. \tag{6.11}$$

According to our previous observation the difference between these two expressions must be of the form $\Delta S^{(2)}$. In fact, we find

$$\Lambda^{(2)} - \Lambda'^{(2)} = \frac{1}{16}\theta^{ij}\theta^{kl}\Big([\partial_i a_k, [b_l, a_j] + [a_l, b_j]] - i[[a_k, b_i], [a_l, a_j]]\Big) = \Delta S^{(2)} \,, \tag{6.12}$$

with $S^{(2)}$ given by

$$S^{(2)} = \frac{1}{16}\theta^{ij}\theta^{kl}[\partial_i a_k, [a_l, a_j]] \,. \tag{6.13}$$

This expression for $S^{(2)}$ can be obtained in the following way. According to the prescriptions in the previous section, before we can apply K to $\Lambda^{(2)} - \Lambda'^{(2)}$, we need that $\Delta(\Lambda^{(2)} - \Lambda'^{(2)})$ vanishes algebraically, without using the relations (5.2). We observe that

$$\Delta(\Lambda^{(2)} - \Lambda'^{(2)}) = -\frac{1}{16}\theta^{ij}\theta^{kl}\{\Delta F_{ik}, a_j b_l + b_j a_l\} = \Delta(\frac{1}{16}[\Delta F_{ik}, a_j a_l]) \tag{6.14}$$

which vanishes only by means of the constraint $\Delta F_{ij} = 0$. Therefore we add to $\Lambda^{(2)} - \Lambda'^{(2)}$ a term $-\frac{1}{16}\theta^{ij}\theta^{kl}[\Delta F_{ik}, a_j a_l]$ and only at this point we apply K, which yields (6.13).

To ensure that $\Delta(\Lambda^{(2)} - \Lambda'^{(2)})$ vanishes, we could have also used our other prescription to symmetrize $\Lambda^{(2)} - \Lambda'^{(2)}$ with respect to all derivatives and then use the substitution (5.11)

$$f_{ij} - D_i a_j - D_j a_i + i[a_i, a_j] = 0 \,, \tag{6.15}$$

to replace F with f

$$\begin{aligned}\Lambda^{(2)} - \Lambda'^{(2)} &= \frac{1}{16}\theta^{ij}\theta^{kl}\Big([F_{ik} + i[a_i, a_k], [b_l, a_j]] + i[[a_k, b_i], [a_j, a_l]]\Big) \\ &= \frac{1}{16}\theta^{ij}\theta^{kl}[f_{ik}, [b_l, a_j]] \,.\end{aligned} \tag{6.16}$$

By applying K we immediately get

$$S^{(2)} = K(\Lambda^{(2)} - \Lambda'^{(2)}) = \frac{1}{32}\theta^{ij}\theta^{kl}[f_{ik}, [a_l, a_j]] \,. \tag{6.17}$$

By substituting back the expression for f_{ik} and noting that

$$\theta^{ij}\theta^{kl}[[a_k,a_i],[a_j,a_l]]=0 \tag{6.18}$$

we again recover (6.13).

By following the same procedure we can compute directly a solution of (6.3) at the second order.

$$\begin{aligned}\Lambda''^{(2)} &= -\frac{1}{2}\theta^{ij}\left\{a_i,\frac{1}{3}D_j\Lambda^{(1)}+\frac{i}{4}[a_j,\Lambda^{(1)}]\right\}+\\ &+\theta^{ij}\theta^{kl}\Bigg(-\frac{i}{16}[D_ia_k,D_jb_l]+[[a_i,a_k],\frac{1}{24}D_jb_l+\frac{i}{32}[a_j,b_l]]+\frac{1}{24}[D_ia_k,[a_j,b_l]]+\\ &\quad+\frac{1}{8}\Bigg(a_i\left(\frac{1}{3}D_ja_k-\frac{1}{3}D_ka_j+\frac{i}{2}[a_j,a_k]\right)b_l-\\ &\quad\quad-b_i\left(\frac{1}{3}D_ja_k-\frac{1}{3}D_ka_j+\frac{i}{2}[a_j,a_k]\right)a_l+\\ &\quad\quad+\left\{\frac{1}{6}(D_ia_k-D_ka_i)+\frac{i}{4}[a_i,a_k],\{a_l,b_j\}\right\}\Bigg)\Bigg).\end{aligned} \tag{6.19}$$

To obtain this result we first observe that

$$\Delta M^{(2)}=\frac{1}{8}\theta^{ij}\theta^{kl}\Big(-2b_i\Delta F_{jk}b_l+b_ib_k\Delta F_{lj}+\Delta F_{ik}b_lb_j\Big)\,. \tag{6.20}$$

Again, as $\Delta M^{(2)}$ vanishes only due to the constraint, we add

$$m^{(2)}=\frac{1}{16}\theta^{ij}\theta^{kl}\Big(2a_i\Delta F_{jk}b_l+2b_i\Delta F_{jk}a_l-(a_ib_k-b_ia_k)\Delta F_{lj}-\Delta F_{ik}(b_la_j-a_lb_j)\Big) \tag{6.21}$$

in such a way as to obtain $\Delta(M^{(2)}+m^{(2)})=0$. Notice that there is an ambiguity in the choice of $m^{(2)}$, but we have chosen the particular $m^{(2)}$ which respects the reality structure, i.e. which provides us with a real Λ''_2. Moreover, observe that

$$\ell\Lambda^{(1)}=0,\quad \ell\partial_i\Lambda^{(1)}=0\;. \tag{6.22}$$

This is a consequence of the fact that

$$\ell K=0 \tag{6.23}$$

and

$$\Lambda^{(1)}=KM^{(1)}\,. \tag{6.24}$$

As expected, the difference between our solution $\Lambda''^{(2)}$ (6.19) and the solution $\Lambda^{(2)}$ (6.10) is again of the form $\Delta S'^{(2)}$ (up to a term which vanishes by the constraint).

$$\begin{aligned}\Lambda''^{(2)}-\Lambda^{(2)} &= \theta^{ij}\theta^{kl}\Big[\Delta\Big(\frac{1}{24}([a_j,[D_ia_k,a_l]]+2(D_ia_ka_ja_l+a_la_jD_ia_k))\Big)+\\ &\quad+\frac{1}{16}[a_ia_k,\Delta F_{jl}]\Big]\;.\end{aligned} \tag{6.25}$$

A similar technique can be followed for the potential A_i. Moreover, if $\Lambda^{(n)}$ is changed by an amount $\Delta S^{(n)}$

$$\Lambda^{(n)} \to \Lambda^{(n)} + \Delta S^{(n)} \tag{6.26}$$

then the corresponding change in the potential is

$$A_i^{(n)} \to A_i^{(n)} + D_i S^{(n)} \,. \tag{6.27}$$

This follows from the fact that the equation of order n for the gauge field is always of the form

$$\Delta A_i^{(n)} = D_i \Lambda^{(n)} + \cdots \,. \tag{6.28}$$

Notice that (6.27) is a consequence of the fact that the coboundary operator Δ commutes with the covariant derivative D_i.

Acknowledgments

We are very grateful to Julius Wess. A seminar he gave in Berkeley on the content of [3] was the original inspiration for our work. This work was supported in part by the Director, Office of Science, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY-95-14797. B.L.C. is supported by the DFG (Deutsche Forschungsgemeinschaft) under grant CE 50/1-1.

A. Expansions of Weyl-Moyal star products

In this appendix we give some useful expressions arising from the expansion of the Weyl-Moyal product.

First, for simplicity, we will define

$$\begin{aligned} \partial_{I_n} &= \partial_{i_1} \cdots \partial_{i_n} \\ \theta^{I_n J_n} &= \theta^{i_1 j_1} \cdots \theta^{i_n j_n} \,. \end{aligned} \tag{A.1}$$

We will expand out $*$-products using Moyal's formula:¡

$$\begin{aligned} f(x) * g(x) &= e^{\frac{i}{2}\theta^{ij}\partial_y\partial_z} f(y) g(z)|_{y=z=x} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2}\right)^n \theta^{i_1 j_1} \cdots \theta^{i_n j_n} \partial i_1 \cdots \partial_{i_n} f(x) \partial_{j_1} \cdots \partial_{j_n} g(x) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2}\right)^n \theta^{I_n J_n} \partial_{I_n} f(x) \partial_{J_n} g(x) \,. \end{aligned} \tag{A.2}$$

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Inserting this expansion into (3.7) and requiring that the equation is satisfied order by order in θ, we find the following expression

$$
\begin{aligned}
\delta_v \Lambda^{(n)} - i\left\{\Lambda^{(n)}, v\right\} &= \Delta\Lambda^{(n)} \\
&= \sum_{p=1}^{n-1}\Bigg(\frac{i}{(n-p)!}\left(\frac{i}{2}\right)^{n-p}\theta^{I_{n-p}J_{n-p}}\left\{\partial_{I_{n-p}}\Lambda_p, \partial_{J_{n-p}}v\right\} + \\
&\qquad + \frac{i}{(p-1)!}\left(\frac{i}{2}\right)^{p-1}\theta^{I_{p-1}J_{p-1}}\sum_{q=1}^{n-p}\partial_{I_{p-1}}\Lambda_q\partial_{J_{p-1}}\Lambda_{n-q-p+1}\Bigg) + \\
&\quad + \frac{i}{n!}\left(\frac{i}{2}\right)^{n}\theta^{I_nJ_n}\left(\partial_{I_n}v\right)\left(\partial_{J_n}v\right).
\end{aligned}
$$

Up to the second order this equation reads

$$0^{th}: \ \delta_v v = iv^2 \tag{A.3}$$

$$1^{st}: \ \Delta\Lambda^{(1)} = -\frac{1}{2}\theta^{ij}b_ib_j \tag{A.4}$$

$$2^{nd}: \ \Delta\Lambda^{(2)} = -\frac{i}{8}\theta^{ij}\theta^{kl}\partial_ib_k\partial_jb_l - \frac{1}{2}\theta^{ij}[b_i, \partial_j\Lambda^{(1)}] + i\Lambda^{(1)}\Lambda^{(1)}\,. \tag{A.5}$$

Analogously the equation (3.8) for the gauge potential A_i

$$\delta_v A_i = \partial_i\Lambda - i\,[A_i \stackrel{\star}{,} \Lambda] \tag{A.6}$$

reads

$$0^{th}: \ \Delta A_i^{(0)} = b_i \tag{A.7}$$

$$1^{st}: \ \Delta A_i^{(1)} = D_i\Lambda^{(1)} - \frac{1}{2}\theta^{kl}\{b_k, \partial_l a_i\} \tag{A.8}$$

$$
\begin{aligned}
2^{nd}: \ \Delta A_i^{(2)} &= D_i\Lambda^{(2)} + i[\Lambda^{(1)}, A_i^{(1)}] - \frac{1}{2}\theta^{kl}\{b_k, \partial_l A_i^{(1)}\} - \\
&\quad - \frac{1}{2}\theta^{kl}\{\partial_k\Lambda^{(1)}, \partial_l a_i\} - \frac{i}{8}\theta^{kl}\theta^{mn}[\partial_k b_m, \partial_l\partial_n a_i]\,.
\end{aligned} \tag{A.9}
$$

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